

# THE BRILL-NOETHER CURVE AND PRYM-TYURIN VARIETIES

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ABSTRACT. We prove that the Jacobian of a general curve  $C$  of genus  $g = 2a + 1$ , with  $a \geq 2$ , can be realized as a Prym-Tyurin variety for the Brill-Noether curve  $W_{a+2}^1(C)$ . As consequence of this result we are able to compute the class of the sum of secant divisors of the curve  $C$ , embedded with a complete linear series  $g_{3a-2}^{a-1}$ .

## 1. INTRODUCTION

Consider a smooth general curve  $C$  of genus  $g \geq 5$ . The locus  $W_d^r(C)$  parametrizing line bundles  $L$  of degree  $d$  over  $C$  with  $h^0(L) \geq r + 1$ , is an irreducible variety of dimension equal to the Brill-Noether number  $\rho = \rho(g, r, d)$ . In particular,  $W_d^r(C)$  is a smooth curve when  $\rho = 1$ . In the case of  $g = 5$ ,  $r = 1$ , the involution  $\iota : L \mapsto \omega_C \otimes L^{-1}$ , induces an automorphism of the curve  $W_4^1(C)$ , which is of genus 11. Since  $C$  is general, the quotient map  $W_4^1(C) \rightarrow W_4^1(C)/\iota$  is an étale double covering over a curve of genus 6. If  $P$  denotes the Prym variety associated to this covering, it is known that  $P$  is isomorphic to the Jacobian  $JC$  as a principally polarized abelian variety ([11]). The main result of this paper shows that this situation generalizes to curves of higher odd genus, obtaining in this way Prym-Tyurin varieties.

Recall that a principally polarized abelian variety (ppav)  $(P, \Xi)$  is a Prym-Tyurin variety if there exists a smooth projective curve  $X$ , such that  $P$  is an abelian subvariety of the Jacobian  $JX$  and the restriction of the principal polarization of  $JX$  to  $P$  is algebraically equivalent to  $e\Xi$ , where  $e \in \mathbb{Z}_{>0}$  is the exponent of  $P$  in  $JX$ . In that case, we say that  $P$  is a Prym-Tyurin variety for the curve  $X$  with exponent  $e$ .

Let  $g = 2a + 1$ , for  $a \geq 2$ . The locus  $W_{a+2}^1(C)$  is a smooth curve, which from now on will be called the *Brill-Noether curve*. We define a correspondence  $\gamma$  on  $W := W_{a+2}^1(C)$ , hence an endomorphism of the Jacobian  $JW$ , by means of the multiplication of sections. More precisely,

$$L \mapsto \gamma(L) := \{L' \in W \mid H^0(L) \otimes H^0(\omega_C \otimes (L')^{-1}) \rightarrow H^0(\omega_C \otimes L \otimes (L')^{-1}) \text{ is not injective}\}.$$

Let  $P := \text{Im}(1 - \gamma) \subset JW_{a+2}^1(C)$ . We prove the following theorem.

**Theorem 1.1.** *Let  $C$  be a general curve of genus  $g = 2a + 1$ . The subvariety  $P := \text{Im}(1 - \gamma)$  is a Prym-Tyurin variety for the Brill-Noether curve  $W_{a+2}^1(C)$  of exponent the Catalan number*

$$\frac{(2a)!}{a!(a+1)!}.$$

*Moreover,  $P \simeq JC$  as principally polarized abelian varieties.*

This result can also be interpreted from the point of view of enumerative geometry. It is reasonable to expect that, under suitable generality assumptions, a linear series  $L \in W_d^r(C)$  has finitely many  $(2r-2)$ -secant  $(r-2)$ -planes that is, divisors  $D \in C^{(2r-2)}$  such that  $h^0(L(-D)) \geq 2$ . In that case the number of secants is computed by the Castelnuovo formula ([1, Chapter VIII]). Then one can associate to every linear series  $g_d^r$  an element of  $\text{Pic}(C)$ , namely the class of the

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sum of the secant divisors. By the results of Ciliberto ([2]) it is natural to expect that this class should depend only on the canonical divisor and the  $g_d^r$ . For instance, when  $\rho(g, r, d) = 0$  one can assign to the curve the class of the sum of the elements in  $W_d^r(C)$ . In this situation, Franchetta's conjecture implies that the sum is a multiple of the canonical bundle (see [4]).

For  $a \geq 4$ , the residual linear system of  $L \in W_{a+2}^1(C)$  defines an embedding  $C \hookrightarrow \mathbb{P}^{a-1}$ , whose image admits finitely many  $(2a-4)$ -secant  $(a-3)$ -planes. These secants are in bijection with the elements of  $\gamma(L)$  by setting  $L' = \omega_C \otimes L^{-1}(-D) \in \gamma(L)$ , where  $D$  is the divisor defined by a secant plane of the embedded curve. As an application of the Theorem 1.1, we are able to determine the class in  $\text{Pic}(C)$  of the sum of the secant divisors.

**Theorem 1.2.** *Let  $C$  be a general smooth curve of genus  $2a+1$ . For any line bundle  $L \in W_{a+2}^1(C)$  we have that*

$$\bigotimes_{L' \in \gamma(L)} L' = \omega_C^\alpha \otimes L^{1-e},$$

where  $e = \frac{(2a)!}{a!(a+1)!}$  and  $\alpha = \frac{a+2}{4a}(g(W) - 1 - e(g-1))$ .

A principally polarized abelian variety can always be realized as a Prym-Tyurin variety for some curve, but with a very large exponent (see [5, Corollary 12.2.4]). In fact, the curve for which a ppav is a Prym-Tyurin variety is not uniquely determined. It is an open problem to find, for a fixed  $g$ , the smallest integer  $m$  such that any ppav of dimension  $g$  is a Prym-Tyurin variety of exponent  $e \leq m$ . For instance, Mumford's results ([12]) show that the general ppav of dimension  $g = 4, 5$  is a Prym-Tyurin variety of exponent 2 for a curve of genus  $2g-1$ . Another example of Prym-Tyurin varieties of small exponent can be found in [10], where the authors exhibit a family of Prym-Tyurin varieties of dimension 6 and exponent 6.

## 2. THE BRILL-NOETHER CURVE

Let  $C$  be a general curve of genus  $g = 2a+1$  satisfying Petri's theorem. Let  $\omega_C$  denote the canonical line bundle on  $C$ . Consider the Brill-Noether locus  $W := W_{a+2}^1(C)$  consisting of line bundles  $L$  on  $C$  of degree  $a+2$ , with  $h^0(C, L) \geq 2$ . Since  $C$  is general and  $\rho(g, 1, a+2) = 1$ , the locus  $W$  is a smooth irreducible curve naturally embedded in  $\text{Pic}^{a+2}(C)$ . The genus of the Brill-Noether curve is computed by the formula ([6, Theorem 4]):

$$(2.1) \quad g(W) = \frac{a}{a+2} \cdot \frac{2g!}{a!(a+1)!} + 1.$$

We fix a point  $L_0 \in \text{Pic}^{a+2}(C)$  and consider the embedding

$$\varphi : W \rightarrow JC, \quad L \mapsto L \otimes L_0^{-1}.$$

**Lemma 2.1.** *The curve  $W$  generates  $JC$  as an abelian group.*

*Proof.* The embedding  $\varphi : W \hookrightarrow JC$  induces a morphism  $\tilde{\varphi} : JW \rightarrow JC$ . It suffices to show that  $\tilde{\varphi}$  is surjective. It has been shown in [9], that the induced map

$$\varphi_* : H_1(W, \mathbb{Z}) \longrightarrow H_1(JC, \mathbb{Z}) \simeq H_1(C, \mathbb{Z})$$

is surjective. This map corresponds to the rational representation of  $\tilde{\varphi}$  and it determines it completely. Hence,  $\tilde{\varphi}$  is surjective.  $\square$

Thus we have a short exact sequence

$$(2.2) \quad 0 \longrightarrow K_{a+2}^1(C) \longrightarrow JW_{a+2}^1(C) \xrightarrow{\tilde{\varphi}} JC \longrightarrow 0,$$

where  $\tilde{\varphi}$  is the map which takes a class of equivalence of divisors of degree zero in  $W_{a+2}^1$  to its linear equivalence class as a divisor on the curve  $C$ . The following result is proved in [3, Theorem 1.1].

**Theorem 2.2.** *For a general curve  $C$  of genus  $g \geq 3$ , the abelian variety  $K_{a+2}^1(C)$  is connected and has no non-trivial endomorphisms which are rationally determined.*

By rationally determined we mean defined over the field of rational functions of  $\mathcal{M}_{g,1}$ , the moduli space of smooth pointed curves of genus  $g$ .

Let us denote  $\theta_C : JC \xrightarrow{\sim} \widehat{JC}$  (respectively  $\theta_W$ ) the principal polarization of  $JC$  (respectively that of  $JW$ ). By dualizing the exact sequence (2.2), we find that

$$\varphi^* = \theta_W^{-1} \circ \hat{\varphi} \circ \theta_C : JC \rightarrow JW,$$

is an embedding since  $\hat{\varphi}$  is also one (see [5, Prop. 2.4.2]). We shall show that the image of  $\varphi^*$  defines an abelian subvariety of  $JW$ , which is a Prym-Tyurin variety for  $W$ . A polarized abelian variety  $(P, \Xi)$  is a Prym-Tyurin variety for a curve  $C$  if there is an embedding  $i_P : P \hookrightarrow JC$  such that  $i_P^* \Xi \equiv e\Xi$ ; the integer  $e$  is called the exponent of  $P$ . We will use Welters' criterion for Prym-Tyurin varieties ([5, Theorem 12.2.2]).

**Theorem 2.3.** (*Welters' Criterion*). *Let  $(P, \Xi)$  be a ppav of dimension  $g$  and  $C$  a smooth curve. Then  $(P, \Xi)$  is a Prym-Tyurin variety of exponent  $e$  for  $C$  if and only if it exists a morphism  $\phi : C \rightarrow P$  such that*

- a)  $\phi^* : P \rightarrow JC$  is an embedding,
- b)  $\phi_*[C] = \frac{e}{(g-1)!} \bigwedge^{g-1} [\Xi]$  in  $H^{2g-2}(P, \mathbb{Z})$ .

**Theorem 2.4.** *The Jacobian  $JC$  is a Prym-Tyurin variety for  $W$  of exponent the Catalan number*

$$(2.3) \quad e = \frac{(2a)!}{a!(a+1)!}.$$

*Proof.* We apply the Criterion 2.3 to the embedding  $\varphi^* : JC \rightarrow JW$ . It suffices to show that  $\varphi_*[W]$  has the required cohomology class. The class of the curve  $W$  in  $H^{2g-2}(\text{Pic}^{a+2}(C), \mathbb{Z})$  is given by ([1, p. 320])

$$(2.4) \quad [W_{a+2}^1(C)] = \frac{1}{a!(a+1)!} \bigwedge^{g-1} [\Theta_C].$$

Hence

$$\varphi_*[W] = \frac{1}{a!(a+1)!} \bigwedge^{g-1} [\Theta_C] = \frac{e}{(2a)!} \bigwedge^{g-1} [\Theta_C]$$

in  $H^{2g-2}(JC, \mathbb{Z})$ . □

### 3. A CORRESPONDENCE ON THE BRILL-NOETHER CURVE

We define the following correspondence on the Brill-Noether curve  $W$ :

$$\gamma : L \mapsto \{L' \in W \mid \mu : H^0(C, L) \otimes H^0(C, \omega_C \otimes (L')^{-1}) \rightarrow H^0(C, \omega_C \otimes L \otimes (L')^{-1}) \text{ is not injective}\},$$

where  $\mu$  denotes the multiplication of sections. It has been shown in [8] that this correspondence is non-empty for any  $a \geq 2$ . The correspondence  $\gamma$  defines an endomorphism (denoted by the same symbol)  $\gamma \in \text{End}(JW)$  by

$$\left[ \sum n_i L_i \right] \mapsto \left[ \sum n_i \gamma(L_i) \right],$$

where  $L_i$  are points on the curve  $W$  (corresponding to line bundles of degree  $a + 2$ ). Using the base-point-free-pencil trick, one checks that  $L' \in \gamma(L)$  if and only if  $H^0(C, \omega_C \otimes L^{-1} \otimes (L')^{-1}) \neq 0$ . So, we can rewrite the correspondence  $\gamma$  as

$$\gamma(L) = \{L' \in W \mid H^0(C, \omega_C \otimes L^{-1} \otimes (L')^{-1}) \neq 0\}.$$

From this description follows that  $\gamma$  is symmetric. Moreover, since  $C$  is general the Gieseker-Petri Theorem ([1, p. 215]) ensures that the multiplication map

$$H^0(C, L) \otimes H^0(C, \omega_C \otimes L^{-1}) \rightarrow H^0(C, \omega_C)$$

is injective for any line bundle  $L \in W_{a+2}^1(C)$ . Thus the correspondence  $\gamma$  has no fixed points, i.e.  $\gamma$  does not intersect the diagonal  $\Delta \subset W \times W$ . This also shows that the induced endomorphism of  $JW$ , is not a multiple of the identity, since these endomorphisms are induced by divisors of the form  $n\Delta$ , for  $n \in \mathbb{Z}$ .

For instance, for  $a = 2$  the correspondence  $\gamma$  induces an involution on the curve  $W$  of genus 11, namely  $\iota : L \mapsto \omega_C \otimes L^{-1}$ . It is known that the corresponding Prym variety associated to the étale double covering  $W \rightarrow W/\iota$  is an abelian subvariety of  $JW$  of dimension 5 isomorphic to the Jacobian of  $C$  ([11]).

**Lemma 3.1.** *Let  $a \geq 3$  and  $g = 2a + 1$ . The degree of the correspondence  $\gamma$  is given by the Castelnuovo number*

$$(3.1) \quad C(a-1, 3a-2, 2a+1) = \sum_{i=0}^{a-2} \frac{(-1)^i}{a+2} \binom{a}{a-2-i} \binom{2a-i}{a-1-i}.$$

*Proof.* Let  $L \in W_{a+2}^1(C)$  and set  $M = \omega_C \otimes L^{-1} \in W_{3a-2}^{a-1}(C)$ . An element  $L' \in W_{a+2}^1(C)$  is in  $\gamma(L)$  if and only if  $H^0(M \otimes (L')^{-1}) \neq 0$ , that is, if  $M \otimes (L')^{-1} = \mathcal{O}_C(D)$  for an effective divisor  $D$  of degree  $2a - 4$ . So  $L'$  is of the form  $M(-D)$  with  $h^0(M(-D)) \geq 2$ . Hence the degree is given by the degree of the degeneracy locus in  $C_d$  (the  $d$ -symmetric power of  $C$ ) of the divisors of degree  $d = 2a - 4$  that impose at most  $d - r = a - 2$  conditions on  $|M|$ . Thus one can interpret the degree of  $\gamma$  as the number of the  $(2a - 4)$ -secant  $(a - 3)$ -planes in the linear system  $|M|$ . For the general curve there are finitely many of such  $(a - 3)$ -planes ([7]) and their number is given by the Castelnuovo formula ([1, Chapter VIII]).  $\square$

The endomorphism  $\gamma$  also defines a map  $W_{a+2}^1(C) \rightarrow \text{Pic}^m(C)$ , where  $m := (a + 2)(\deg \gamma)$  by considering  $\gamma(L)$  as a tensor product of line bundles on  $C$ . More precisely, if  $D_i \in \text{Div}^{2a-4}(C)$ , for  $i = 1, \dots, \deg \gamma$ , are the secant divisors of the image of  $C$  in  $|\omega_C \otimes L^{-1}|^* \simeq \mathbb{P}^{a-1}$ , we set  $L_i := M(-D_i) \in \text{Pic}^{a+2}(C)$  and  $\gamma(L)$  can be viewed as the line bundle

$$(3.2) \quad \bigotimes_{i=1}^{\deg \gamma} L_i \in \text{Pic}^m(C).$$

We denote  $\mathcal{P}$  the Zariski open subset consisting of all equivalence classes  $(C, x) \in \mathcal{M}_{g,1}$ , with  $C$  a curve having no non-trivial automorphisms and satisfying the Petri condition. There exists a smooth scheme  $\mathcal{G}_d^r$  and a morphism  $\mathcal{G}_d^r \rightarrow \mathcal{P}$  such that the fiber over any closed point  $t \in \mathcal{P}$  is isomorphic to  $G_d^r(C)$ , the variety parametrizing all the  $g_d^r$ 's on  $C$ . For  $a \geq 4$ , set  $r = 1$ ,  $d = a + 2$  and  $\mathcal{G} := \mathcal{G}_{a+2}^1$ . Now, let  $\mathcal{H}$  be the Hilbert scheme of curves of degree  $3a - 2$  and of genus  $2a + 1$  in  $\mathbb{P}^{a-1}$  and  $\mathcal{H}_1$  the open set of a component of  $\mathcal{H}$  with general moduli, parametrizing curves without nontrivial automorphisms. For every point  $z = ((C, x), D) \in \mathcal{G}$  denote by  $\Gamma \subset \mathbb{P}^{a-1}$  the image of  $C$  by the residual series  $|\omega_C \otimes L^{-1}|$ , with  $L = \mathcal{O}_C(D)$  and by  $[\Gamma]$  the corresponding point in  $\mathcal{H}_1$ . Let  $\mathcal{H}_z$  be a closed subset of  $\mathcal{H}_1$  given by the orbit of  $[\Gamma]$  under the action of  $PGL(a, \mathbb{C})$  (see [2, §3]). The map  $\gamma : z \mapsto \gamma(L)$  induces a regular section  $\mathcal{H}_z \rightarrow \text{Pic}^m(C)$ . By varying the

curve  $C$  we obtain a rationally determined line bundle  $\mathcal{L}$  on the universal family  $\mathcal{F}$  over  $\mathcal{H}_1$ , such that the restriction of  $\mathcal{L}$  to the fiber over  $z$  is isomorphic to  $\gamma(L) \in \text{Pic}^m(C)$ , where  $\gamma(L)$  is the tensor product (3.2). As a consequence of the Theorem 2.2 one has the following result ([3, Theorem 1.2]).

**Theorem 3.2.** *Let  $\mathcal{H}$  be any component of the Hurwitz scheme of coverings of  $\mathbb{P}^1$  of degree  $d$  and genus  $g \geq 3$  containing curves with general moduli and with  $\rho = 1$ . Then the group of rationally determined line bundles of the universal family over  $\mathcal{H}$  is generated by the relative canonical bundle and the hyperplane bundle.*

It follows that there exist integers  $\alpha, \beta$  such that

$$(3.3) \quad \gamma(L) = \omega_C^{\otimes \alpha} \otimes L^{\otimes \beta} \in \text{Pic}^m(C).$$

We are able to deduce the coefficients  $\alpha$  and  $\beta$  as an application of Theorem 3.3. Set  $P := \text{Im}(1 - \gamma) \subset JW$ . On the light of the Theorem 2.2, one does not expect other subvarieties of  $JW$ , other than the obvious ones. More precisely, we prove:

**Theorem 3.3.** *The subvariety  $P$  is isomorphic to  $JC$ . In particular,  $P$  is a Prym-Tyurin variety for  $W$  of exponent  $e$ .*

*Proof.* Consider the map  $\tilde{\varphi}|_P : P \rightarrow JC$  and suppose it is non-zero. Then by Theorem 2.2,  $\tilde{\varphi}|_P$  is an isogeny. The embedding  $\varphi^* : JC \rightarrow JW$  gives then an isomorphism  $JC \simeq P$ . In particular,  $(P, \Theta_{C|_P})$  is a Prym-Tyurin variety of exponent  $e$  for  $W$ . If the restriction of  $\tilde{\varphi}$  to  $P$  is zero, the complementary subvariety  $Z$  of  $P$  with respect to  $\Theta_W$  is isogenous to  $JC$ , via the restriction  $\tilde{\varphi}|_Z : Z \rightarrow JC$ . In this case  $\varphi^*(JC) = Z$  and  $Z$  is a Prym-Tyurin of exponent  $e$  for  $W$ . Moreover,  $Z = \text{Im}(e - 1 + \gamma)$ . Using the formula in [5, Corollary 5.3.10], one computes that

$$\dim Z = \frac{(e - 1)g(W) + \deg \gamma}{e}.$$

Since  $\dim Z = \dim JC = g$ , we have  $(e - 1)g(W) + \deg \gamma = eg$ . By Lemma 3.4 we obtain that

$$(e - 2)g(W) = -2 \deg \gamma,$$

which is a contradiction since  $e \geq 2$  and  $\deg \gamma > 0$ . Therefore  $JC \simeq \text{Im}(1 - \gamma)$ .  $\square$

**Lemma 3.4.** *The equation  $g(W) - \deg(\gamma) = eg$  holds.*

*Proof.* A direct computation.  $\square$

#### 4. THE EQUIVALENCE CLASS OF THE SUM OF SECANTS TO A CURVE

For any line bundle  $L \in W_{a+2}^1(C)$ , consider the product

$$\gamma(L) = \bigotimes_{i=1}^{\deg \gamma} L_i$$

as defined in §3.

**Theorem 4.1.** *Let  $C$  be a general smooth curve of genus  $2a + 1$ . For any line bundle  $L \in W_{a+2}^1(C)$  we have that*

$$\gamma(L) = \bigotimes_{i=1}^{\deg \gamma} L_i = \omega_C^\alpha \otimes L^{1-e},$$

where  $e = \frac{(2a)!}{a!(a+1)!}$  and  $\alpha = \frac{a+2}{4a}(g(W) - 1 - e(g - 1))$ .

*Proof.* The norm-endomorphism corresponding to the subvariety  $P \subset JW$  is  $1 - \gamma$ . It satisfies  $(1 - \gamma)^2 = e(1 - \gamma)$ , or equivalently, the quadratic equation

$$(4.1) \quad (1 - e - \gamma)(1 - \gamma) = \gamma^2 + (e - 2)\gamma - (e - 1) = 0$$

on the Jacobian  $JW$ . Consider the projection  $\tilde{\varphi}$  from  $P = \text{Im}(1 - \gamma)$  to  $JC$ . Fix  $M \in W_{a+2}^1(C)$ . Then, by (3.3), there exists an integer  $\beta$  such that

$$\tilde{\varphi}(1 - \gamma)(L - M) = L \otimes M^{-1} \otimes L^{-\beta} \otimes M^\beta.$$

Since the relation (4.1) holds on  $JC$  as well, we obtain

$$\begin{aligned} (1 - e - \gamma)(L \otimes M^{-1})^{\otimes 1-\beta} &= (L \otimes M^{-1})^{\otimes (1-e)(1-\beta)} \otimes (L \otimes M^{-1})^{\otimes -(1-\beta)\beta} \\ &= (L \otimes M^{-1})^{\otimes (1-\beta)(1-e-\beta)} \\ &= \mathcal{O}_C \end{aligned}$$

for all  $L \in W$ . Therefore  $(1 - \beta)(1 - e + \beta) = 0$ . If  $\beta = 1$ ,  $\tilde{\varphi}(1 - \gamma) = 0$ , which is a contradiction to the fact that  $\tilde{\varphi}|_P$  is surjective. Hence  $\beta = 1 - e$ . In order to compute the value of  $\alpha$  one compares the degrees in the equation (3.3) and uses Lemma 3.4.  $\square$

For example, for a general line bundle  $L \in W_5^1(C)$  on a curve  $C$  of genus 7, the image of the map  $\phi : C \rightarrow |\omega_C \otimes L^{-1}|^*$  is a plane curve with 8 nodes. Let  $p_i, q_i$ , for  $i = 1, \dots, 8$ , denote the pre-images of the nodes. Set  $M := \omega_C \otimes L^{-1} \in W_7^2(C)$ . Hence

$$\gamma(L) = \bigotimes_{i=1}^8 M(-p_i - q_i) \in \text{Pic}^{40}(C).$$

By the adjunction formula we have that

$$\omega_C = M^4(-\sum_{i=1}^8 p_i + q_i),$$

that is,

$$\bigotimes_{i=1}^8 M(-p_i - q_i) = \omega_C^5 \otimes L^{-4},$$

which is predicted by Theorem 4.1 since the Catalan number is equal to 5. A less trivial example is the case of a general curve  $C$  of genus 9 embedded in  $\mathbb{P}^3$  by the linear system  $|M|$ , with  $M \in W_{10}^3(C)$ . The space curve admits 43 4-secant lines, the genus of the curve  $W_{10}^3(C)$  is 169 and the exponent of the Prym-Tyurin variety is 14. Let us denote  $D_i \in \text{Div}^4(C)$  the corresponding divisors. By Theorem 4.1 we obtain  $\alpha = 21$ ,  $\beta = -13$  and

$$\bigotimes_{i=1}^{43} \mathcal{O}_C(D_i) = M^{30} \otimes \omega_C^{-8}.$$

**Remark 4.2.** For a general curve  $C$ , the subring of  $H^*(C_2, \mathbb{Q})$  generated by the fundamental classes of algebraic cycles on  $C_2$  is generated by the class of a fiber of the projection  $\pi_1 : C_2 \rightarrow C$  and the diagonal ([1, p. 359]). In the situation of the Brill-Noether curve, such subring of  $H^*(W_2, \mathbb{Q})$  has an extra generator induced by the correspondence  $\gamma \subset W \times W$ .

**Remark 4.3.** It would be interesting to study the properties of the curve  $W$ , for instance determine its gonality or if it has a special Brill-Noether behavior.

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